Extension of Berenger's PML for Bi-Isotropic Media

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***Abstract*—An extension of Berenger's perfectly matched layer (PML) absorbing boundary conditions to adapt bi-isotropic media is formulated. The method is developed under simple algebraic conditions and validated with the finite-difference time-domain method (FDTD).**

***Index Terms*—Absorbing boundary conditions, bi-isotropic media, FDTD method, perfectly matched layer.**

I. Introduction

THE propagation of waves in bi-isotropic media has become a topic of interest during recent years [1]. Starting from previous extensions of Berenger's perfectly matched layer (PML) absorbing boundary conditions [2] for anisotropic media [3], [4], in this letter we propose a PML medium (BiPML) to adapt bi-isotropic media. It has been validated and tested through an implementation of the finite-difference time-domain method (FDTD) to simulate monochromatic wave propagation in these media.

II. Maxwell's Curl Equations in Bi-Isotropic Media

Waves propagating in source-free bi-isotropic media can be decomposed into a sum of left- and right-hand elliptical polarized waves [1] of the form

$$\vec{\Psi}(\vec{r},t) = \vec{\Psi}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \tag{1}$$

with a complex amplitude $\vec{\Psi}_0$, frequency ω, and propagating in the $\hat{n}$ direction with wavevector $\vec{k} = k\hat{n}$, which is assumed to be real (lossless media) for simplicity. For these waves, Maxwell's curl equations can be written as [1]

$$-j\omega\mu\vec{H} - j\omega\zeta\vec{E} = -j\tilde{K}\cdot\vec{E} \tag{2}$$

$$\omega\varepsilon\vec{E} + j\omega\xi\vec{H} = -j\tilde{K}\cdot\vec{H} \tag{3}$$

where $\vec{E} = (E_x, E_y, E_z)^t$, $\vec{H} = (H_x, H_y, H_z)^t$, with the superindex ^{t} indicating the transposition of matrices. $\tilde{K}$ is the wavevector matrix given by

$$\tilde{K} = \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix}. \tag{4}$$

ε and μ are the real dielectric and magnetic constants, and ξ and ζ are the parameters describing the bi-isotropic behavior of the medium. These latter can be expressed in terms of the real Tellegen (χ) and chirality (κ) parameters [1] through $\xi = (\chi - j\kappa)/c$ and $\zeta = (\chi + j\kappa)/c$, with $c = 1/\sqrt{\mu_0\varepsilon_0}$ being the free-space light speed.

The dispersion relation is obtained by enforcing a nontrivial value of the solution of (2) and (3), obtaining two wavenumbers for each frequency: one for the right-hand polarization (k_+) and the other for the left-hand polarization (k_-): $k_\pm = (\omega/c)(\cos\theta \pm \kappa_r)$, with $\sin\theta = \chi_r \equiv (\chi/\sqrt{\varepsilon_r\mu_r})$, $\kappa_r \equiv (\kappa/\sqrt{\varepsilon_r\mu_r})$, and ε_r and μ_r the relative permittivity and permeability of the material, respectively.

III. Equations for the BiPML Medium

Following Berenger's ideas [2], together with the general method presented in [3] and [4], the BiPML medium is defined such that the six-component *split* fields $\vec{h}_s$ and $\vec{e}_s$ in this medium are

$$\vec{e}_s \equiv (e_{xy}, e_{xz}, e_{yz}, e_{yx}, e_{zx}, e_{zy})^t \tag{5}$$

$$\vec{h}_s \equiv (h_{xy}, h_{xz}, h_{yz}, h_{yx}, h_{zx}, h_{zy})^t \tag{6}$$

related to the three-component *compact* fields $\vec{e} \equiv (e_x, e_y, e_z)^t$ and $\vec{h} \equiv (h_x, h_y, h_z)^t$ through

$$\vec{e} = \tilde{C}\cdot\vec{e}_s, \quad \vec{h} = \tilde{C}\cdot\vec{h}_s, \quad \tilde{C} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{7}$$

Let us assume that the propagation into the BiPML of the split fields can also be decomposed into functions of the form

$$\vec{\Psi}_s(\vec{r},t) = \vec{\Psi}_{so} e^{j(\omega t - \vec{\gamma}\cdot\vec{r})} \tag{8}$$

with $\vec{\Psi}_{so}$ being the complex amplitude and $\vec{\gamma}$ the complex wavevector in the BiPML medium. These functions will be made to fulfill equations similar to (2) and (3), but including anisotropic lossy terms as in [2], as well as a bianisotropic behavior through the use of two complex matrices $\tilde{A}$ and $\tilde{A}^*$, in the following manner:

$$-j\omega\mu\tilde{I}_6\cdot\vec{h}_s - \tilde{\sigma}^*\cdot\vec{h}_s - j\omega\zeta\tilde{A}^*\cdot\vec{e}_s = -j\tilde{G}_s\cdot\vec{e}_s \tag{9}$$

$$j\omega\varepsilon\tilde{I}_6\cdot\vec{e}_s + \tilde{\sigma}\cdot\vec{e}_s + j\omega\xi\tilde{A}\cdot\vec{h}_s = -j\tilde{G}_s\cdot\vec{h}_s \tag{10}$$

where $\tilde{I}_n$ is the nth-order identity matrix, $\tilde{G}_s$ the split wavevector matrix

$$\tilde{G}_s = \tilde{g}_s\cdot\tilde{C}, \quad \tilde{g}_s = \begin{pmatrix} 0 & 0 & \gamma_y \\ 0 & -\gamma_z & 0 \\ \gamma_z & 0 & 0 \\ 0 & 0 & -\gamma_x \\ 0 & \gamma_x & 0 \\ -\gamma_y & 0 & 0 \end{pmatrix} \tag{11}$$

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$\tilde{\sigma}$ is the (6×6) real diagonal matrix

$$\tilde{\sigma} = \begin{pmatrix} \sigma_y & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_z & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_z & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_x & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_x & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_y \end{pmatrix} \quad (12)$$

and $\tilde{\sigma}^*$ is a matrix with a similar form.

Finally, matrix $\tilde{A}$ is given by

$$\tilde{A} = \tilde{a} \cdot \tilde{C}, \quad \tilde{a} = \begin{pmatrix} a_y & 0 & 0 \\ a_z & 0 & 0 \\ 0 & a_z & 0 \\ 0 & a_x & 0 \\ 0 & 0 & a_x \\ 0 & 0 & a_y \end{pmatrix} \quad (13)$$

and similarly for $\tilde{A}^* = \tilde{a}^* \cdot \tilde{C}$. All the elements of these matrices will be obtained in the next section by imposing perfect matching of the BiPML and the bi-isotropic medium.

IV. Perfect Matching

If a wave propagating in the bi-isotropic medium is incident upon an interface with a BiPML medium, no reflection will occur at the interface for any angle of incidence, nor for any frequency, if there is a perfect matching between both media. A sufficient condition is obtained by enforcing the continuity of the components of the field in the original medium with those of the compact field in the BiPML. For instance, for the interface plane $z = 0$

$$(\vec{E} = \vec{e})_{z=0}, \quad (\vec{H} = \vec{h})_{z=0}. \quad (14)$$

The continuity condition implies [3], [4]:

1) phase continuity $(e^{j\vec{k}\cdot\vec{r}} = e^{j\vec{\gamma}\cdot\vec{r}})_{z=0} \Rightarrow \gamma_x = k_x, \gamma_y = k_y$, with no restriction on γ_z;
2) amplitude continuity $\vec{E}_0 = \vec{e}_0, \vec{H}_0 = \vec{h}_0$.

In order to impose condition 2), let us first extract $\vec{h}_{so}$ and $\vec{e}_{so}$ from (9) and (10), respectively, and obtain the compact fields $\vec{e}_o$ and $\vec{h}_o$ through (7). Then, extracting $\vec{H}_o$ and $\vec{E}_o$ from (2) and (3), we finally obtain a system of equations which is fulfilled for any $\vec{E}_0$ and $\vec{H}_0$ if

$$\tilde{C} \cdot \left(\tilde{I}_6 - j\frac{\tilde{\sigma}^*}{\omega\mu}\right)^{-1} \cdot \tilde{g}_s = \tilde{C} \cdot \left(\tilde{I}_6 - j\frac{\tilde{\sigma}}{\omega\varepsilon}\right)^{-1} \cdot \tilde{g}_s = \tilde{K} \quad (15)$$

and

$$\tilde{C} \cdot \left(\tilde{I}_6 - j\frac{\tilde{\sigma}^*}{\omega\mu}\right)^{-1} \cdot \tilde{a}^* = \tilde{C} \cdot \left(\tilde{I}_6 - j\frac{\tilde{\sigma}}{\omega\varepsilon}\right)^{-1} \cdot \tilde{a} = \tilde{I}_3. \quad (16)$$

The first equality of (15) holds if $\tilde{\sigma}$ and $\tilde{\sigma}^*$ are related by the usual Berenger condition $(\tilde{\sigma}/\varepsilon) = (\tilde{\sigma}^*/\mu)$. Taking this into account, and given that $\tilde{C} \cdot \tilde{C}^t = 2\tilde{I}_3$, (16) holds if $\tilde{a} = \frac{1}{2}(\tilde{I}_6 - j(\tilde{\sigma}/\omega\varepsilon)) \cdot \tilde{C}^t = \tilde{a}^* = \frac{1}{2}(\tilde{I}_6 - j(\tilde{\sigma}^*/\omega\mu)) \cdot \tilde{C}^t$.

At $z = 0$ the phase continuity 1), together with the last equality of (15), leads to $\sigma_x = \sigma_y = 0$ and $\gamma_z = k_z(1 - j(\sigma_z/\omega\varepsilon))$.

Solving the dispersion relation for the BiPML medium governed by (9) and (10), with the above values for $\tilde{\sigma}, \tilde{\sigma}^*$, $\tilde{a}$ and $\tilde{a}^*$, it is found that the BiPML, in fact, supports attenuated waves with wavevector $\vec{\gamma} = (k_x, k_y, k_z(1 - j(\sigma_z/\omega\varepsilon)))$.

Notice that the BiPML has turned out to be a bianisotropic PML medium, which is reduced to Berenger's PML when the Tellegen and chirality parameters of the original medium are null.

V. Implementation and Validation

With the aim of testing the matching conditions with FDTD, we need an explicit-in-time difference scheme. For this, time domain versions of (2), (3), (9), and (10), valid for harmonic fields, have been used. For instance for (3)

$$\varepsilon\frac{\partial\vec{E}}{\partial t} = \nabla \times \vec{H} - \frac{\omega\kappa}{c}\vec{H} - \frac{\chi}{c}\frac{\partial\vec{H}}{\partial t} \quad (17)$$

and for (10)

$$\begin{aligned} \varepsilon\tilde{I}_6 \cdot \frac{\partial\vec{e}_s}{\partial t} &+ \tilde{\sigma} \cdot \vec{e}_s \\ &= \tilde{R}_s \cdot \vec{h}_s - \frac{1}{2c}\left(\frac{\chi}{\varepsilon}\tilde{\sigma} + \omega\kappa\tilde{I}_6\right) \cdot \tilde{C}^t \cdot \tilde{C} \cdot \vec{h}_s \\ &\quad - \frac{1}{2c}\left(\chi\tilde{I}_6 - \frac{\kappa\tilde{\sigma}}{\omega\varepsilon}\right) \cdot \tilde{C}^t \cdot \tilde{C} \cdot \frac{\partial\vec{h}_s}{\partial t} \end{aligned} \quad (18)$$

with

$$\tilde{R}_s = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ 0 & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial z} & 0 & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & 0 & 0 \\ -\frac{\partial}{\partial y} & -\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \end{pmatrix} \quad (19)$$

where it has been assumed that the values of all the parameters are known at a given frequency ω.

The FDTD algorithm has been simply obtained by approximating the time derivative of $\vec{E}$ and $\vec{e}_s$ in (17) and (18) by second-order finite-centered differences, and the time derivatives of $\vec{H}$ and $\vec{h}_s$ by the following second-order backward formula:

$$\frac{\partial f}{\partial t}(t_0) = \frac{3f(t_0) - 4f(t_0 - \Delta t) + f(t_0 - 2\Delta t)}{2\Delta t}. \quad (20)$$

The rest of the discretizations in (17) and (18) have been performed using a second-order two-node scheme through finite centered differences and means [5]. The ratio between the spatial increment and the time increment $(\Delta/\Delta t) = 4\ c$ has been found to provide stable results for the simulations shown here.

Since the BiPML is just a generalization of Berenger's PML medium, all the details concerning the implementation of the BiPML (corners, conductivity spatial profile, numerical measurement of the reflection coefficient, etc.) are treated in the same way as in [2].

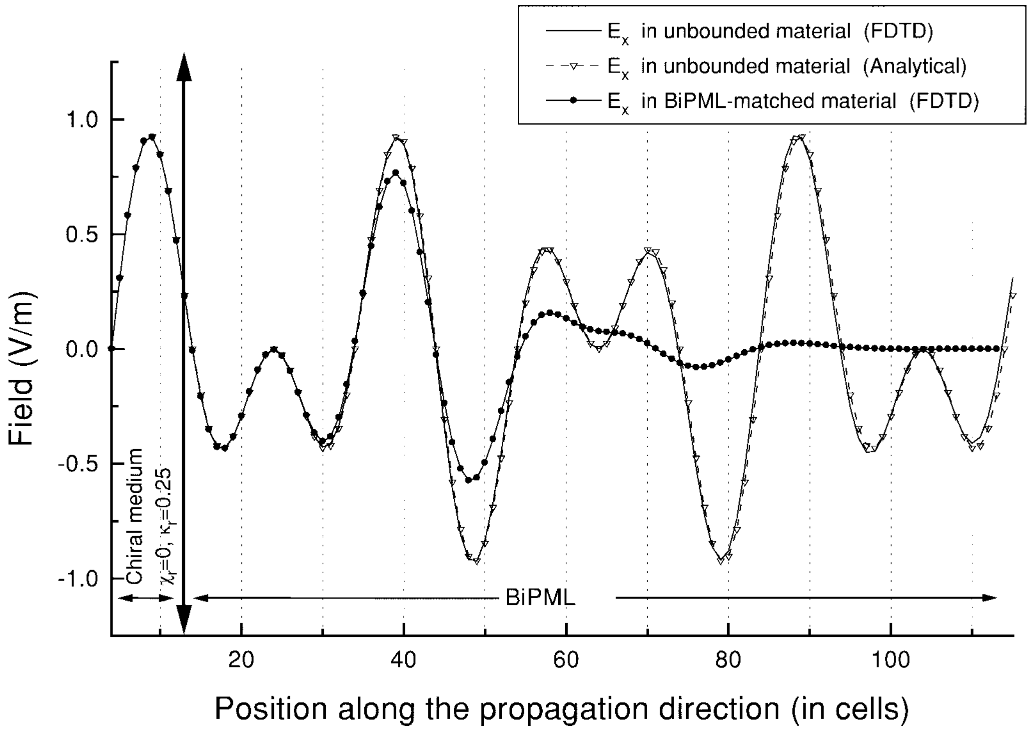

Fig. 1. Comparison of field profiles at the time step $n = 600$ for a chiral medium with $\varepsilon_r = 1, \mu_r = 1$, $\chi_r = 0$, and $\kappa_r = 0.25$, adapted with a 100-cell BiPML.

Fig. 1 shows the spatial profile of the E_x component of a monochromatic wave propagating in a two-dimensional (2-D) chiral medium with $\varepsilon_r = 1$, $\mu_r = 1, \chi_r = 0$, and $\kappa_r = 0.25$. Such a wave is composed of a right- and a left-hand elliptical polarized plane wave, with respective wavelengths $\lambda_+ = 16$ and $\lambda_- = 26.67$ cells, which is normally incident with the BiPML. In order to show the damping of the propagated field, a 100-cell-depth BiPML with a parabolic conductivity profile was chosen. This provided a normal theoretical reflection coefficient of -136 and -82 dB for the right- and left-hand polarizations, respectively. Results of -110 and -76 dB were numerically found for the above reflection coefficient at a point placed one cell before the interface.

Analytical results for propagation in the unbounded original medium are also compared to FDTD results in Fig. 1. The slight differences between the two results, when the distance from the source point increases, are due to the dispersion inherent to the FDTD procedure.

Another 2-D bi-isotropic medium was also simulated with $\varepsilon_r = 1$, $\mu_r = 1, \chi_r = 0.66$ and $\kappa_r = 0.25$, $\lambda_+ = 20$, and $\lambda_- = 40$ cells, adapted by a 12-cell-depth BiPML medium. This provided a normal theoretical reflection coefficient of -131 and -65 dB for the right- and left-hand polarizations, respectively. Experimentally, -62 and -54 dB were found for the reflection coefficient at a point placed one cell before the interface. As expected, better agreement between the theoretical and numerical reflection coefficients is found in the 100-layer case, due to numerical reflections in each change of conductivity from layer to layer, more abrupt in the 12-cell case than in the 100-cell case.

VI. Conclusions

In this letter an extension of the PML absorbing boundary conditions has been developed for bi-isotropic media. We have found simple algebraic conditions that guarantee the continuity of all the field components at the interface, which is sufficient for perfect matching. A time-domain version of the equations, valid for harmonic fields, has been discretized with a FDTD method in order to test the conditions for 2-D bi-isotropic media.

References

[1] I. V. Lindell, A. H. Sihvola, S. A. Tretyakov, and A. J. Vitanen, *Electromagnetic Waves in Chiral and Bi-Isotropic Media*. Norwood, MA: Artech House, 1994.

[2] J.-P. Berenger, “A perfectly matched layer for the absorption of electromagnetic waves,” *J. Comput. Phys.*, no. 114, pp. 185–200, 1994.

[3] I. Villó Pérez, S. González García, R. Gómez Martín, and B. Garc´ía Olmedo, “Extension of Berenger’s Absorbing Boundary Conditions to match dielectric anisotropic media,” *IEEE Microwave Guided Wave Lett.*, vol. 7, pp. 302–304, Sept. 1997.

[4] ——, “Generalization of Berenger’s absorbing boundary conditions for 3D magnetic and dielectric anisotropic media,” *Microwave Opt. Technol. Lett.*, vol. 18, no. 2, pp. 126–130, 1998.

[5] S. González García, T. Materdey Hung-Bao, R. Gómez Martín, and B. García Olmedo, “On the application of finite methods in time domain to anisotropic dielectric waveguides,” *IEEE Trans. Microwave Theory Tech.*, vol. 44, pp. 2195–2206, Dec. 1996.